THE INDEX OF REDUCIBILITY OF PARAMETER IDEALS IN LOW DIMENSION

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Abstract. In this paper we present results concerning the following question: If M is a finitely-generated module with finite local cohomologies over a Noetherian local ring $(A, \mathfrak{m})$, does there exist an integer ℓ such that every parameter ideal for M contained in $\mathfrak{m}^\ell$ has the same index of reducibility? We show that the answer is *yes* if $\dim M = 1$ or if $\dim M = 2$ and $\operatorname{depth} M > 0$. This research is closely related to work of Goto-Suzuki and Goto-Sakurai; Goto-Sakurai have supplied an answer of *yes* in case M is Buchsbaum.

1. Introduction

Let A be a d-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, and let M be a finitely generated A-module. Recall that a submodule of M is called *irreducible* if it cannot be written as the intersection of two larger submodules. It is well known that every submodule N of M can be expressed as an irredundant intersection of irreducible submodules, and that the number of irreducible submodules appearing in such an expression depends only on N and not on the expression [ShV, p. 92-3].

For an ideal I of A, we say that I is *cofinite* on M if the module M/IM has finite length. For an ideal I of A which is cofinite on M, the *index of reducibility* of I on M is defined as the number of submodules appearing in an irredundant expression of IM as an intersection of irreducible submodules of M. We denote the index of reducibility of I on M by $\mathrm{N}_A(I; M)$.

The smallest number of generators of an ideal which is cofinite on M is the dimension of M, and a cofinite ideal having this minimal number of generators is called a *parameter ideal* for M. Our interest in the index of reducibility of parameter ideals stems from the relationship with the Cohen-Macaulay and Gorenstein properties. In 1956, D. G. Northcott proved that in a Cohen-Macaulay local ring, the index of reducibility of any parameter ideal depends only on the ring [N, Theorem 3]. This result extends to modules, and the common index of reducibility of parameter ideals for a Cohen-Macaulay module M is called the *(Cohen-Macaulay) type* of M. We denote the type of M by $\mathrm{r}_A(M)$.

As a partial converse, Northcott along with D. Rees proved in [NR1] that if every parameter ideal of A is irreducible, then A is Cohen-Macaulay. This provides an attractive characterization of a Gorenstein local ring as a local ring in which every parameter ideal is irreducible.

One might suspect that the Cohen-Macaulay property of a local ring is characterized by the constant index of reducibility of parameter ideals. However, in 1964,

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S. Endo and M. Narita in [EN] gave an example of a Noetherian local ring in which the index of reducibility of each parameter ideal is two, and yet the ring is not Cohen-Macaulay.

In 1984, S. Goto and N. Suzuki generalized the example of Endo-Narita, as well as undertaking a study of the supremum of the index of reducibility of parameter ideals for M [GSu]. We refer to this supremum as the *Goto-Suzuki* type (*GS-type*) of M, and denote it by $\overline{r}_A(M)$. In the case where M is Cohen-Macaulay, the GS-type $\overline{r}_A(M)$ is equal to the type $\mathrm{r}_A(M)$. However, Goto-Suzuki provide examples where the GS-type of a Noetherian local ring is infinity.

We introduce some terminology in order to state one of the main results of Goto-Suzuki. We denote the ith local cohomology module of M with respect to $\mathfrak{m}$ by $\mathrm{H}^i_{\mathfrak{m}}(M)$. We say that M has *finite local cohomologies* if the modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ have finite length for $i \neq d$. We use $\lambda_A(M)$ to denote the length of M, and we set $\mathfrak{S}(M) = \lambda_A(\mathrm{Hom}_A(k, M))$, the *socle dimension* of M. We let $\mu_A(M)$ denote the minimal number of generators of M, and let E denote the injective hull of the residue field k. The main result of Goto-Suzuki concerning the GS-type of a module having finite local cohomologies is the following:

Theorem 1.1 (Goto-Suzuki)**.** *Let M be a finitely generated d-dimensional A-module with finite local cohomologies. Then we have the following inequalities:*

$$\sum_{i=0}^{d} \binom{d}{i} \mathfrak{S}\left(\mathrm{H}^i_{\mathfrak{m}}(M)\right) \leq \overline{r}_A(M) \leq \sum_{i=0}^{d-1} \binom{d}{i} \lambda_A\left(\mathrm{H}^i_{\mathfrak{m}}(M)\right) + \mu_{\hat{A}}(K), \tag{1.1}$$

where $\hat{A}$ is the $\mathfrak{m}$-adic completion of A and $K = \mathrm{Hom}_A\left(\mathrm{H}^d_{\mathfrak{m}}(M), E\right)$ is the canonical module of the completion of M.

Proof. See [GSu, Theorems 2.1 and 2.3]. □

In 1994, Kawasaki used this result to determine conditions under which a module having finite GS-type is Cohen-Macaulay. The main result of Kawasaki in [K] is that if A is the homomorphic image of a Cohen-Macaulay ring, then M is Cohen-Macaulay if and only if $\overline{r}_A(M)$ is finite, $M_{\mathfrak{p}}$ is Cohen-Macaulay for all primes in the support of M with $\dim M_{\mathfrak{p}} < \overline{r}_A(M)$, and all the associated primes of M have the same dimension.

In two recent papers [GS1, GS2], Goto with H. Sakurai has returned to the study of the index of reducibility of parameter ideals in order to investigate when the equality $I^2 = QI$ holds for a parameter ideal Q in A, where $I = (Q :_A \mathfrak{m})$. According to earlier research of A. Corso, C. Huneke, C. Polini, and W. Vasconcelos [CHP, CP, CPV], this equality holds for all parameter ideals Q in case A is a Cohen-Macaulay ring which is not regular. Goto-Sakurai generalize this and say that if A is a Buchsbaum ring whose multiplicity is greater than 1, then the equality $I^2 = QI$ holds for any parameter ideal whose index of reducibility is the GS-type of A. Thus, if A is a Buchsbaum ring whose multiplicity is greater than 1 and if A has constant index of reducibility of parameter ideals, then the equality $I^2 = QI$ holds for all parameter ideals Q.

Most pertinent to our discussion is Corollary 3.13 of Goto-Sakurai in [GS1], which states that if A is a Buchsbaum ring of positive dimension, then there is an integer ℓ such that the index of reducibility of Q is independent of Q and equals

$\overline{r}_A(A)$ for all parameter ideals $Q \subseteq \mathfrak{m}^\ell$. In view of this, it is natural to ask the following question:

Question 1.2. *Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring having finite local cohomologies. Is there an integer* ℓ *such that the index of reducibility of any parameter ideal contained in* $\mathfrak{m}^\ell$ *is the same?*

We note that if A is a ring satisfying the hypothesis of the question, and if the answer to the question is *yes* for A, then the common index of reducibility of parameter ideals in high powers of the maximal ideal is equal to the lower bound of Goto-Suzuki: $\sum_{i=0}^{d} \binom{d}{i} \mathfrak{S}\left(\mathrm{H}^i_{\mathfrak{m}}(A)\right)$. This is because implicit in the proof of the lower bound [GSu, Theorem 2.3] we find that given any system of parameters x_1, $\ldots$, x_d for A, there are integers n_i such that the parameter ideal $(x_1^{n_1}, \ldots, x_d^{n_d})A$ has index of reducibility $\sum_{i=0}^{d} \binom{d}{i} \mathfrak{S}\left(\mathrm{H}^i_{\mathfrak{m}}(A)\right)$.

The main result of the current paper is the following:

Theorem 1.3. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and let* M *be a finitely-generated* A*-module of dimension* $d \leq 2$. *Suppose either* M *has dimension 1, or* M *has finite local cohomologies and depth at least one. Then there exists an integer* ℓ *such that for every parameter ideal* $\mathfrak{q}$ *for* M *contained in* $\mathfrak{m}^\ell$, *the index of reducibility of* $\mathfrak{q}$ *on* M *is independent of* $\mathfrak{q}$ *and is given by*

$$\mathrm{N}_A(\mathfrak{q}; M) = \sum_{i=0}^{d} \binom{d}{i} \mathfrak{S}\left(\mathrm{H}^i_{\mathfrak{m}}(M)\right). \tag{1.2}$$

Proof. See Theorem 2.3 and 3.3. □

As a corollary of this result we prove that a Noetherian local ring A of dimension at most 2 is Gorenstein if and only if A has finite local cohomologies, and inside every power of the maximal ideal of A there exists an irreducible parameter ideal. A Noetherian local ring having the property that every power of its maximal ideal contains an irreducible parameter ideal is called an *approximately Gorenstein* ring, or is said to have *small cofinite irreducibles (SCI)*. The author's original motivation for studying the index of reducibility of parameter ideals comes from questions that arose while studying M. Hochster's paper [H] exposing the relationship between modules having SCI and the condition that cyclic purity implies purity.

We present an example of a complete Noetherian local ring of dimension d and depth $d-1$ ($d > 1$), such that $\mathrm{H}^{d-1}_{\mathfrak{m}}(A)$ is not finitely generated, and such that in every power of the maximal ideal there is a parameter ideal with index of reducibility 2 and a parameter ideal with index of reducibility 3. This example is obtained as an idealization, so we present several basic results relating the index of reducibility with an idealization. Using a result of C. Lech [L], we are able to obtain such an example among Noetherian local domains.

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2. Background and Dimension One

We begin with some terminology.

Definitions and Notation 2.1. Let A denote a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, let I denote an ideal of A, and let M denote a finitely generated A-module.

(1) The *socle* of M is defined to be $\mathrm{S}(M) = (0 :_M \mathfrak{m})$. Note that $\mathrm{S}(M) \cong \mathrm{Hom}_A(k, M)$. The socle is naturally a vector space over k, and we denote its dimension by $\mathfrak{S}(M)$.

(2) Suppose M has depth t. The *type* of M is defined by
$$\mathrm{r}_A(M) = \dim_k \mathrm{Ext}_A^t(k, M).$$
Note that if I is cofinite on M, then $\mathrm{N}_A(I; M) = \mathrm{r}_A(M/IM)$, and if M has depth 0, then $\mathfrak{S}(M) = \mathrm{r}_A(M)$. See [BH, p. 13] for more information on the type of a module.

(3) Given an A-algebra B, we define $M_B = M \otimes_A B$. Then $(A/I)_B \cong B/IB$, and if B is flat over A, then for any submodule N of M, we have $M_B/N_B \cong (M/N)_B$.

(4) We say that an ideal J of A is a *reduction* of I if $J \subseteq I$ and there is some integer n such that $I^{n+1} = JI^n$. We say that J is a *minimal reduction* of I if there is no reduction of I properly contained in J. The *reduction number* of I with respect to J is defined as
$$\mathrm{red}_J(I) = \min\{n : I^{n+1} = JI^n\}.$$
We define the *reduction number* of I to be
$$\mathrm{red}(I) = \min\{\mathrm{red}_J(I) : J \text{ is a minimal reduction of } I\}.$$

We must reduce to the case that the residue field is infinite in order to obtain a principal reduction of the maximal ideal. Thus our first result is a basic lemma concerning the behavior of the index of reducibility under a flat, local change of base. We omit the proof of Lemma 2.2, but cite [Na, Theorem 19.1], [BH, Proposition 1.2.16] and [M, Theorem 7.4] for relevant related results.

Lemma 2.2. *Suppose A and B are Noetherian local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively, let M be a finitely generated A-module, and suppose B is a flat A-algebra for which $\mathfrak{n} = \mathfrak{m}B$. Let I be an ideal of A and let N be a submodule of M.*

(1) *If M has finite length, then M_B has finite length, and $\lambda_B(M_B) = \lambda_A(M)$.*
(2) *If I is an ideal of A which is cofinite on M, then IB is cofinite on M_B.*
(3) *If I is an ideal of A which is cofinite on M, then*
$$\mathrm{N}_B(IB; M_B) = \mathrm{N}_A(I; M).$$
(4) $(N :_M I)_B \cong (N_B :_{M_B} IB)$.
(5) $\mathrm{H}^0_{\mathfrak{n}}(M_B) = \mathrm{H}^0_{\mathfrak{m}}(M)_B$.

Now we are prepared to prove that parameter ideals in high powers of a one dimensional Noetherian local ring all have the same index of reducibility.

Theorem 2.3. *Suppose A is a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module of dimension 1. Set $W = \mathrm{H}^0_{\mathfrak{m}}(M)$. There is an integer ℓ such that the index of reducibility of any parameter ideal $\mathfrak{q} \subseteq \mathfrak{m}^\ell$ is independent of $\mathfrak{q}$ and is given by*
$$\mathrm{N}_A(\mathfrak{q}; M) = \mathfrak{S}(M) + \mathrm{r}_A(M/W).$$

When $M = A$ and k is infinite, the integer ℓ may be taken to be

$$\ell = \max\{c, d\} + 1$$

where $c = \mathrm{red}\,(\mathfrak{m})$ and d is the smallest integer with $\mathfrak{m}^d \cap W = 0$.

Proof. First we show that it suffices to prove the theorem in the case where the residue field k is infinite. Suppose B is any local flat A-algebra whose maximal ideal $\mathfrak{n}$ is extended from A and whose residue field is infinite (for instance, $B = A[x]_{\mathfrak{m}A[x]}$ for an indeterminate x). Then according to Theorem A.11 on p. 415 of [BH], the B-module M_B has dimension 1. Therefore we may apply the theorem to M_B; let ℓ be the integer guaranteed by the theorem. Set $V = \mathrm{H}^0_{\mathfrak{n}}\,(M_B)$; then according to Part 5 of Lemma 2.2, we have $V = W_B$.

Suppose $\mathfrak{q}$ is a parameter ideal for M that is contained in $\mathfrak{m}^\ell$. Then $\mathfrak{q}B$ is a parameter ideal for M_B which is contained in $\mathfrak{n}^\ell$, so we have

$$\mathrm{N}_B\,(\mathfrak{q}B; M_B) = \mathfrak{S}\,(M_B) + \mathrm{r}_B\,(M_B/V)\,. \tag{2.1}$$

According to Part 3 of Lemma 2.2, the left side of equality 2.1 equals $\mathrm{N}_A\,(\mathfrak{q}; M)$, so it remains to see that $\mathfrak{S}\,(M_B) = \mathfrak{S}\,(M)$ and $\mathrm{r}_B\,(M_B/V) = \mathrm{r}_A\,(M/W)$. For the first equality, if $W = 0$ then $V = 0$ and we have nothing to show. Otherwise, M has depth 0 and thus according to Proposition 1.2.16 part (a) from [BH], so does M_B, so by part (b) of the same Proposition, we have

$$\mathfrak{S}\,(M_B) = \mathrm{r}_B\,(M_B) = \mathrm{r}_A\,(M) = \mathfrak{S}\,(M)\,.$$

To see the equality $\mathrm{r}_B\,(M_B/V) = \mathrm{r}_A\,(M/W)$, we first note that

$$M_B/V = M_B/W_B \cong (M/W)_B.$$

Hence, by another application of Proposition 1.2.16 part (b) from [BH], we have

$$\mathrm{r}_B\,(M_B/V) = \mathrm{r}_B\,((M/W)_B) = \mathrm{r}_A\,(M/W)\,.$$

Thus we see that we may assume the residue field k is infinite.

Our second task is to see that we may replace the ring A by the ring $C = A/I$, where I is the annihilator of M. Let $\mathfrak{q}$ denote an ideal of A which is a parameter ideal for M. The A-module M becomes an C-module in a natural way, and the submodule structure of M remains unchanged. The maximal ideal of A extends to that of C, the residue field of C is k, and $\mathfrak{q}C$ is an ideal of C which is a parameter ideal for M. If N is any finitely generated A-module annihilated by I, then as sets we have $\mathrm{Hom}_A\,(k, N) = \mathrm{Hom}_C\,(k, N)$, and we easily check that this equality is actually an isomorphism of A-modules. Thus $\mathfrak{S}\,(N)$ does not depend on whether we view N as an A-module or a C-module. Since the numbers $\mathrm{N}_A\,(\mathfrak{q}; M)$, $\mathfrak{S}\,(M)$, and $\mathrm{r}_A\,(M/W)$ are all calculated as socle dimensions of quotients of M, they do not change when we view M as an C-module. Thus we replace A by C and assume that A has Krull dimension one.

If $W = 0$, then M is Cohen-Macaulay, $\mathfrak{S}\,(M) = 0$, and for a parameter ideal $\mathfrak{q} = aA$ we have

$$\mathrm{N}_A\,(\mathfrak{q}; M) = \mathfrak{S}\,(M/aM) = \dim_k \mathrm{Ext}^1_A\,(k, M) = \mathrm{r}_A\,(M)\,. \tag{2.2}$$

Thus the proof is complete in this case.

Now suppose $W \neq 0$. Since W has finite length and

$$\bigcap_{i=1}^{\infty} (\mathfrak{m}^n M \cap W) = 0,$$

there is some integer d with $\mathfrak{m}^d M \cap W = 0$.

Since the residue field is infinite, $\mathfrak{m}$ has a principal reduction; i.e., there is an element $x \in \mathfrak{m}$ and an integer $c \geq 1$ such that $\mathfrak{m}^{c+1} = x\mathfrak{m}^c$ ([BH, Corollary 4.6.10, p. 191]). Set $\ell = \max\{c, d\} + 1$. Then we have arrived at a situation where any parameter a for M which is in $\mathfrak{m}^\ell$ is of the form $a = xy$ with $y \in \mathfrak{m}^d$. Furthermore, since we have assumed M is faithful, the parameters for M are just the parameters for A [E, Proposition 10.8, p. 237]. Since the parameters for the one-dimensional ring A are those elements not in any minimal prime ideal of A, we see that if $a = xy$ is a parameter for M, then so are x and y.

Since $aM \cap W = 0$, we have $(W + aM)/aM \cong W$, so, as in the proof of [GSu, Proposition (2.4)], we see that the top row in the following commutative diagram is exact:

(2.3)
$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] & \frac{M}{aM} \arrow[r, "\beta"] & \frac{M}{aM+W} \arrow[r] & 0 \\
& & & \frac{M}{xM+W} \arrow[ul, "f"] \arrow[u, "g"'] &
\end{tikzcd}$$

The maps f and g are each given by multiplication by y. The map f is well-defined since $y(xM+W) = aM + yW = aM$, and g is just the composition $g = \beta f$.

To see that g is injective, view g as multiplication by y as follows:

$$\frac{M/W}{x(M/W)} \xrightarrow{g = \cdot y} \frac{M/W}{a(M/W)}$$

Since M/W is a Cohen-Macaulay module and y is a parameter for M, and thus for M/W, we have that y is regular on M/W. Hence

$$\big(xy(M/W) :_{M/W} y\big) = x(M/W),$$

so that g is injective.

Information on the dimension of the socles is obtained by applying the functor $\operatorname{Hom}_A(k, -)$; what results is the following exact commutative diagram, where the maps induced by f and g are still injective:

(2.4)
$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(k, W) \arrow[r] & \operatorname{Hom}_A\left(k, \frac{M}{aM}\right) \arrow[r, "\beta^*"] & \operatorname{Hom}_A\left(k, \frac{M}{aM+W}\right) \\
& & & \operatorname{Hom}_A\left(k, \frac{M}{xM+W}\right) \arrow[ul, "\operatorname{Hom}_A(k,f)"'] \arrow[u, "\operatorname{Hom}_A(k,g)"']
\end{tikzcd}$$

Here we use β^* to denote $\operatorname{Hom}_A(k, \beta)$.

The important point is that since x and a are still parameters for the Cohen-Macaulay module M/W, we have

$$\begin{aligned}
\dim_k \operatorname{Hom}_A\left(k, \frac{M}{aM+W}\right) &= \dim_k \operatorname{Hom}_A\left(k, \frac{M/W}{a(M/W)}\right) \\
&= \mathrm{N}_A(aA; M/W) \\
&= \mathrm{r}_A(M/W),
\end{aligned}$$

and similarly

$$\dim_k \operatorname{Hom}_A\left(k, \frac{M}{aM+W}\right) = \mathrm{r}_A\left(M/W\right).$$

The map $\operatorname{Hom}_A(k, g)$ is an injection of k-vector spaces, each of dimension $\mathrm{r}_A(M/W)$, hence this map is an isomorphism. From the surjectivity of this map, it follows that β^* is surjective, so that the top row of Diagram 2.4 is exact. Since the length of the middle module is $\mathrm{N}_A(aA; M)$ and the left-hand module is the socle of W, which is the socle of M, we complete the proof in dimension one using the additivity of length. □

3. Dimension 2

We thank the referee for pointing out the fascinating technique in this section, and for pointing out the recent work of Goto-Sakurai [GS1, GS2]. Goto-Sakurai successfully apply this technique to the case of Buchsbaum local rings of arbitrary dimension [GS1, Corollary 3.13].

Suppose $\mathfrak{q} = (x_1, \dots, x_d)A$ is an ideal generated by a system of parameters x_1, $\dots$, x_d for a finitely-generated module M of dimension d over a Noetherian local ring $(A, \mathfrak{m})$. We may form a direct system of modules by setting $M_i = M/(x_1^i, \dots, x_d^i)M$ and defining maps from $M_i \to M_{i+1}$ given by multiplication by $x_1 \cdot \cdots \cdot x_d$. It is known that the direct limit of this system is $\mathrm{H}^d_{\mathfrak{m}}(M)$ [BH, Theorem 3.5.6]. From this point of view, we see that there is a canonical homomorphism from $M/\mathfrak{q}M$ to $\mathrm{H}^d_{\mathfrak{m}}(M)$.

For an ideal I of a Noetherian local ring $(A, \mathfrak{m})$ and a finitely-generated module M we define $\mathrm{U}(I; M)$ to be the unmixed component of the submodule IM of M; that is, $\mathrm{U}(I; M)$ is the intersection of the primary components of the submodule IM whose associated primes have maximal dimension, equal to $\dim M/IM$.

In the course of our proof for dimension 2 we will need to mention several generalizations of the notion of a regular sequence. These definitions can be found in the appendix of [SV], which is a good source for information concerning modules having finite local cohomologies.

Definition 3.1. Suppose $(A, \mathfrak{m})$ is a Noetherian local ring, M is a finitely-generated A-module of dimension $d > 0$, and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. A system of elements $x_1, \dots, x_r$ is called an $\mathfrak{a}$-*weak* M-*sequence* if

$$((x_1, \dots, x_{i-1})M :_M x_i) \subseteq ((x_1, \dots, x_{i-1})M :_M \mathfrak{a})$$

for all $i = 1, \dots, r$.

Let $x_1, \dots, x_d$ be a system of parameters for M and let $\mathfrak{q} = (x_1, \dots, x_d)A$. We say that $x_1, \dots, x_d$ is a *standard system of parameters* of M if $x_1^{n_1}, \dots, x_d^{n_d}$ is a $\mathfrak{q}$-weak M-sequence for all $n_1, \dots, n_d \geq 1$.

We say that $\mathfrak{a}$ is a *standard ideal* with respect to M if every system of parameters of M contained in $\mathfrak{a}$ is a standard system of parameters of M.

We begin by isolating a general statement concerning the index of reducibility of parameter ideals in the case where M has finite local cohomologies.

Proposition 3.2. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and let M be a finitely-generated d-dimensional A-module with $d > 0$ such that M has finite local cohomologies. There exists an integer ℓ such that for every parameter ideal $\mathfrak{q} = (x_1, \dots, x_d)$*

of M, if $\mathfrak{q} \subseteq \mathfrak{m}^\ell$ then the index of reducibility of $\mathfrak{q}$ on M is given by

$$\mathrm{N}_A(\mathfrak{q}; M) = \mathfrak{S}\left(\sum_{i=1}^{d} \frac{U_i + x_i M}{\mathfrak{q}M}\right) + \mathfrak{S}\left(\mathrm{H}^d_{\mathfrak{m}}(M)\right), \tag{3.1}$$

where $U_i = U((x_1, \ldots, \widehat{x_i}, \ldots, x_d)A; M)$.

Proof. Since M has finite local cohomologies, we have a standard ideal $\mathfrak{a}$ for M [SV, Corollary 18, p. 264]; hence every system of parameters of M contained in $\mathfrak{a}$ is an $\mathfrak{a}$-weak M-sequence [SV, Theorem 20, p. 264]. Thus, given any system of parameters $x_1, \ldots, x_d$ of M contained in $\mathfrak{a}$ and any integer $n \geq 1$, we have by [SV, Lemma 23]

$$\begin{aligned}
&\left((x_1^{n+1}, \ldots, x_d^{n+1})M :_M (x_1 \cdot \cdots \cdot x_d)^n\right) \\
&\qquad = (x_1, \ldots, x_d)M + \sum_{i=1}^{d} \left((x_1, \ldots, \widehat{x_i}, \ldots, x_d)M :_M \mathfrak{a}\right).
\end{aligned} \tag{3.2}$$

We note that this right hand side is equal to

$$\sum_{i=1}^{d} \left(((x_1, \ldots, \widehat{x_i}, \ldots, x_d)M :_M x_i) + x_i M\right). \tag{3.3}$$

Set $U_i = \mathrm{U}((x_1, \ldots, \widehat{x_i}, \ldots, x_d)A; M)$. Since $U_i = ((x_1, \ldots, \widehat{x_i}, \ldots, x_d)M :_M x_i)$, we have that

$$\left((x_1^{n+1}, \ldots, x_d^{n+1})M :_M (x_1 \cdot \cdots \cdot x_d)^n\right) = \sum_{i=1}^{d} (U_i + x_i M). \tag{3.4}$$

An important component of this proof is that according to [GS1, Lemma 3.12], we may choose ℓ large enough so that for any ideal $\mathfrak{q}$ generated by a system of parameters for M, if $\mathfrak{q} \subseteq \mathfrak{m}^\ell$ then the canonical map $\phi : M/\mathfrak{q}M \to \mathrm{H}^d_{\mathfrak{m}}(M)$ is surjective on the socles; that is, $\mathrm{Hom}_A(k, \phi)$ is surjective. We also require ℓ to be large enough so that $\mathfrak{m}^\ell \subseteq \mathfrak{a}$.

Let $\mathfrak{q} = (x_1, \ldots, x_d)$ be a parameter ideal of M contained in $\mathfrak{m}^\ell$ and set $U_i = \mathrm{U}((x_1, \ldots, \widehat{x_i}, \ldots, x_d)A; M)$. Let K denote the kernel of the canonical map ϕ from $M/\mathfrak{q}M$ to $\mathrm{H}^d_{\mathfrak{m}}(M)$. According to the definition of the direct limit, we have

$$K = \frac{\cup_{n \geq 1} \left((x_1^{n+1}, \ldots, x_d^{n+1})M :_M (x_1 \cdot \cdots \cdot x_d)^n\right)}{\mathfrak{q}M}; \tag{3.5}$$

thus by Equation 3.4 we see that

$$K = \sum_{i=1}^{d} \frac{U_i + x_i M}{\mathfrak{q}M}. \tag{3.6}$$

Now all that is left is to apply the socle functor $\mathrm{Hom}_A(k, -)$ to the exact sequence

$$0 \to K \to M/\mathfrak{q}M \to \mathrm{H}^d_{\mathfrak{m}}(M) \tag{3.7}$$

and use the surjectivity on the socles to obtain the result. □

Theorem 3.3. *Suppose $(A, \mathfrak{m})$ is a Noetherian local ring and let M be a finitely-generated A-module of dimension 2 such that M has positive depth, and $\mathrm{H}^1_{\mathfrak{m}}(M)$*

is finitely generated. Then there exists an integer ℓ such that for every parameter ideal $\mathfrak{q}$ of M, if $\mathfrak{q} \subseteq \mathfrak{m}^\ell$ then the index of reducibility of $\mathfrak{q}$ on M is given by

$$\mathrm{N}_A(\mathfrak{q}; M) = 2 \cdot \mathfrak{S}\left(\mathrm{H}^1_{\mathfrak{m}}(M)\right) + \mathfrak{S}\left(\mathrm{H}^2_{\mathfrak{m}}(M)\right). \tag{3.8}$$

In particular, parameter ideals for M in large powers of $\mathfrak{m}$ all have the same index of reducibility.

Proof. We begin by obtaining an integer ℓ from Proposition 3.2. As in the proof of that proposition, we may assume that ℓ is large enough so that every system of parameters of M contained in $\mathfrak{m}^\ell$ is a standard system of parameters of M.

Let a, b be a system of parameters of M contained in $\mathfrak{m}^\ell$ and set $\mathfrak{q} = (a, b)A$. Then we have

$$\mathrm{N}_A(\mathfrak{q}; M) = \mathfrak{S}\left(\frac{U_a + bM}{\mathfrak{q}M} + \frac{U_b + aM}{\mathfrak{q}M}\right) + \mathfrak{S}\left(\mathrm{H}^2_{\mathfrak{m}}(M)\right), \tag{3.9}$$

where $U_a = \mathrm{U}(aA; M) = (aM :_M b)$, and similarly for U_b.

Since M has positive depth, a and b are regular elements on M. According to [SV, Theorem and Definition 17, p. 261], a and b both kill $\mathrm{H}^1_{\mathfrak{m}}(M)$. Thus from the long exact sequence for local cohomology obtained from the short exact sequence

$$0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0 \tag{3.10}$$

we see that $\mathrm{H}^0_{\mathfrak{m}}(M/aM) \cong \mathrm{H}^1_{\mathfrak{m}}(M)$. Since M/aM has dimension 1, $\mathrm{H}^0_{\mathfrak{m}}(M/aM)$ is just $\mathrm{U}(aA; M)/aM = (aM :_M b)/aM$. Furthermore, we have a surjective homomorphism $(aM :_M b) \to ((aM :_M b) + bM)/bM$ whose kernel is $(aM :_M b) \cap bM$. Using the definition of a standard system of parameters, we see that this last expression is just aM. Thus we have seen that $\mathrm{H}^1_{\mathfrak{m}}(M) \cong ((aM :_M b) + bM)/bM$. The same holds if we interchange a and b.

At this point all that remains is to see that the sum

$$\frac{U_a + bM}{\mathfrak{q}M} + \frac{U_b + aM}{\mathfrak{q}M} \tag{3.11}$$

is direct. To this end, we note that

$$\begin{aligned}(U_a + bM) \cap (U_b + aM) &= (U_a \cap (U_b + aM)) + bM \\ &= (U_a \cap U_b) + aM + bM\end{aligned} \tag{3.12}$$

Using the fact that a and b are regular on M and form a standard system of parameters of M, we see that

$$U_a \cap U_b = (aM :_M b) \cap (bM :_M a) = aM \cap bM. \tag{3.13}$$

Thus

$$(U_a + bM) \cap (U_b + aM) = \mathfrak{q}M, \tag{3.14}$$

the sum is direct, and our proof is complete. □

Corollary 3.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension at most 2. Then A is Gorenstein if and only if A has finite local cohomologies and every power of the maximal ideal contains an irreducible parameter ideal.*

Proof. If A is Gorenstein, then all the local cohomology modules other than $\mathrm{H}^2_{\mathfrak{m}}(A)$ are zero, and all the parameter ideals are irreducible.

For the other direction, note that it suffices to show that A is Cohen-Macaulay, since a Cohen-Macaulay local ring with an irreducible parameter ideal is precisely a

Gorenstein local ring. The theorem is trivial in dimension zero: a zero dimensional Noetherian local ring is Gorenstein if and only if the zero ideal is irreducible. When the dimension of A is one, the result essentially goes back to Northcott-Rees [NR1, Lemma 7]: If A has SCI, then A has positive depth.

In dimension 2, using the fact that the depth is positive, we know from Theorem 3.2 that all parameter ideals in a high power of the maximal ideal have the same index of reducibility, namely $2 \cdot \mathfrak{S}\left(\mathrm{H}^1_{\mathfrak{m}}(A)\right) + \mathfrak{S}\left(\mathrm{H}^2_{\mathfrak{m}}(A)\right)$. According to our hypothesis, this integer must be 1. Since $\mathrm{H}^2_{\mathfrak{m}}(A)$ is a nonzero Artinian module, it has a nonzero socle. Thus is has a cyclic socle, and $\mathrm{H}^1_{\mathfrak{m}}(A)$ is an Artinian module with zero socle. Thus $\mathrm{H}^1_{\mathfrak{m}}(A)$ is zero, so that A is Cohen-Macaulay. □

Remark 3.5. The technique in this section can be used to give a different proof in the case of dimension 1.

4. An Example

We begin with a lemma concerning idealizations.

Definition 4.1. Given a ring R and an R-module M we define the ring $R \ltimes M$ to be the symmetric algebra of M modulo the square of the positive piece. Thus $R \ltimes M$ is a graded ring whose degree 0 piece is R, whose degree 1 piece is M, and whose components of degree greater than 1 are zero. This ring is called the *idealization* of R with M, or the *trivial extension* of R by M.

For more information on idealization, we refer the reader to [Na, p. 18] or [BH, Exercise 3.3.22].

Lemma 4.2. (1) *Let R be a ring and let M be an R-module. If I is an ideal of R, then*
$$\frac{R \ltimes M}{I(R \ltimes M)} \cong \frac{R}{I} \ltimes \frac{M}{IM}.$$

(2) *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let M be an R-module. Set $A = R \ltimes M$. Then*
$$\mathrm{S}(A) = \mathrm{S}(R) \cap \mathrm{ann}_R(M) + \mathrm{S}(M).$$
In particular, we have
$$\mathfrak{S}(A) = \mathfrak{S}(M) + \dim_k(\mathrm{S}(R) \cap \mathrm{ann}_R(M)).$$

(3) *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let M be an R-module. Set $A = R \ltimes M$. If $\mathfrak{q}$ is an irreducible $\mathfrak{m}$-primary ideal of R which does not contain $\mathrm{ann}_R(M)$, then $\mathfrak{q}A$ is a parameter ideal of A for which*
$$\mathrm{N}_A(\mathfrak{q}A; A) = \mathrm{N}_R(\mathfrak{q}; M) + 1.$$

Proof. (1) As R-modules, we have the isomorphism, and we immediately see that this homomorphism respects multiplication.

(2) We regard A as $A = R + Mt$ with t an indeterminate and $t^2 = 0$. The maximal ideal of A is $\mathfrak{m} + Mt$. An element $r + mt$ is in $\mathrm{S}(A)$ if and only if
$$0 = (r + mt)(\mathfrak{m} + Mt) = r\mathfrak{m} + (rMt + \mathfrak{m}mt),$$
which happens precisely when $r \in \mathrm{S}(R) \cap \mathrm{ann}_R(M)$ and $m \in \mathrm{S}(M)$, as desired. The statement about dimensions now follows from the fact that the k-vector space structure of the socle is induced through its graded R-module structure.

(3) We have

$$\begin{aligned}
\mathrm{N}_A(\mathfrak{q}A;A) &= \mathfrak{S}(A/\mathfrak{q}A)\\
&= \mathfrak{S}\left(\frac{R}{\mathfrak{q}} \ltimes \frac{M}{\mathfrak{q}M}\right)\\
&= \mathfrak{S}(M/\mathfrak{q}M) + \dim_k \mathrm{S}(R/\mathfrak{q}) \cap \mathrm{ann}_{R/\mathfrak{q}}(M/\mathfrak{q}M)\\
&= \mathrm{N}_R(\mathfrak{q};M) + \dim_k \mathrm{S}(R/\mathfrak{q}) \cap \mathrm{ann}_{R/\mathfrak{q}}(M/\mathfrak{q}M)
\end{aligned}$$

Now since $\mathfrak{q}$ is an irreducible $\mathfrak{m}$-primary ideal, the socle of $R/\mathfrak{q}$ is simple and essential, and is thus contained in every nonzero submodule of $R/\mathfrak{q}$. Hence all that remains is to note that $\mathrm{ann}_{R/\mathfrak{q}}(M/\mathfrak{q}M)$ is not zero, since it contains $(\mathfrak{q} + \mathrm{ann}_R(M))/\mathfrak{q}$.

□

We thank the referee for suggesting the following example in dimension 2, and for pointing out the interesting paper of C. Lech [L].

Example 4.3. Let k be a field and let $d > 1$ be an integer. Put $R = k[[x, y, z_3, \ldots, z_d]]$, the formal power series ring over k in d variables. Let $M = R/x^2R$ and set $A = R \ltimes M$. Let w be a new variable; then $A \cong T/(x^2w, w^2)T$, where $T = k[[x, y, z_3, \ldots, z_d, w]]$. Thus A is a complete Noetherian local ring of dimension d and depth $d - 1$.

For each integer $n \geq 3$ we define two parameter ideals of R:

$$\mathfrak{q} = (x^n, y^n, z_3^n, \ldots, z_d^n)R \quad \text{and} \quad \mathfrak{q}' = ((x+y)^n, xy^{n-1}, z_3^n, \ldots, z_d^n)R.$$

Let $Q = \mathfrak{q}A$ and $Q' = \mathfrak{q}'A$; these are parameter ideals of A contained in $\mathfrak{m}^n$. We show that $\mathrm{N}_A(Q;A) = 2$ while $\mathrm{N}_A(Q';A) = 3$.

Neither Q nor Q' contain $\mathrm{ann}_R(M) = x^2R$, and since R is regular (hence Gorenstein), each of $\mathfrak{q}$ and $\mathfrak{q}'$ are irreducible. Hence from Part 3 of Lemma 4.2 we see that to decide the index of reducibility of Q and Q', we just need to consider

$$\mathrm{N}_R(\mathfrak{q};M) = \dim_k R/(\mathfrak{q} + x^2R),$$

and similarly for $\mathfrak{q}'$.

We have

$$\mathfrak{q} + x^2R = (x^2, y^n, z_3^n, \ldots, z_d^n)R \quad \text{and} \quad \mathfrak{q}' + x^2R = (x^2, xy^{n-1}, y^n, z_3^n, \ldots, z_d^n)R.$$

The first of these ideals is a parameter ideal for the Gorenstein ring R, and is thus irreducible. For the second we have

$$(x^2, xy^{n-1}, y^n, z_3^n, \ldots, z_d^n)R = (x^2, y^{n-1}, z_3^n, \ldots, z_d^n)R \cap (x, y^n, z_3^n, \ldots, z_d^n)R.$$

Hence $\mathrm{N}_A(Q;A) = 1 + \mathrm{N}_R(\mathfrak{q};M) = 2$ while $\mathrm{N}_A(Q';A) = 1 + \mathrm{N}_R(\mathfrak{q}';M) = 3$.

Remark 4.4. We would like to point out that a bad example such as this may be obtained among local Noetherian domains. The referee kindly brought our attention to a paper of C. Lech [L] which proves that a complete Noetherian local ring A is the completion of a Noetherian local domain if and only if the prime ring of A is a domain that acts without torsion on A, and the maximal ideal of A is not associated to 0. (Here the prime ring of a ring is the subring generated by the multiplicative identity.)

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